Geophysical Journal International

Geophys. J. Int. (2025) **243,** 1–5
Advance Access publication 2025 September 17
Research Paper

https://doi.org/10.1093/gji/ggaf366

The Gutenberg–Richter law strikes back: the exponentiality of magnitudes is confirmed by worldwide seismicity

Matteo Taroni

National Institute of Geophysics and Volcanology (INGV), Rome 00143, Italy. E-mail: matteo.taroni@ingv.it

Accepted 2025 September 8. Received 2025 September 8; in original form 2025 February 15

SUMMARY

The magnitudes of earthquakes are generally described by an empirical relation called the Gutenberg–Richter law. This relation corresponds to a well-known statistical distribution, that is, the exponential distribution. In this work, we verify the validity of the Gutenberg–Richter law using a 44-yr-long worldwide seismic catalogue of strong ($M_w \geq 6.5$) events, by testing the exponentiality and the independence of the magnitudes. Moreover, we suggest a new way to visualize the distribution of the magnitudes, which complements the classical magnitude frequency distribution plot.

Key words: Statistical methods; Earthquake hazards; Earthquake interaction, forecasting, and prediction.

INTRODUCTION

The Gutenberg–Richter law (Gutenberg & Richter 1944) is one of the most important empirical laws of seismicity; it describes the distribution of earthquake magnitudes using the following formula:

$$\mathrm{Log}_{10}(N) = a - b(M - M_c), \tag{1}$$

where N is the number of events with magnitude $\geq M$, a is the parameter which controls the total number of events, b is the parameter called b-value, which controls the slope of the distribution in a semi-log plot and M_c is the magnitude of completeness of the catalogue. This last parameter is fundamental because the Gutenberg–Richter law is observed only for magnitudes $M \geq M_c$. The magnitude of completeness represents the threshold above which we are (almost) sure of detecting all the seismic events. There are several methods to estimate the magnitude of completeness, and the most used are based on the same assumption for finding the complete part of the catalogue: a goodness-of-fit comparison of the complete part of the catalogue with the Gutenberg–Richter law (Wiemer & Wyss 2000); a maximum-likelihood estimate considering the whole distribution, still assuming for the complete part the Gutenberg–Richter law (Ogata & Katsura 1993); a statistical test for the exponentiality of the magnitudes in the complete part of the catalogue (Herrmann & Marzocchi 2021; Taroni 2023). Indeed, Utsu (1965) and Aki (1965) showed that the Gutenberg–Richter law implies an exponential distribution for the magnitudes, with the unique parameter for such statistical distribution equal to $ln(10)$. However, all these methodologies lead to a sort of circular reasoning: the estimation of the magnitude of completeness is performed by assuming that the catalogue is complete if the Gutenberg–Richter/exponentiality of the magnitudes is matched, but this condition is valid only if the catalogue is complete. Thus, a fair independent test of the reliability of the Gutenberg–Richter law must be performed without assuming the same law in the calculation of the completeness magnitude. Although global seismicity was already been deeply investigated in terms of the magnitude frequency distribution, for example, Kagan (2002), Bird & Kagan (2004), Kagan *et al.* (2010), in this work we want to use the global CMT catalogue (Dziewonski *et al.* 1981; Ekström *et al.* 2012) from 1980 to 2023 (44 yr) to validate the Gutenberg-Richter/exponentiality hypothesis for the magnitude distribution. The novelty of our approach is the independence between the magnitude of completeness and the magnitude distribution estimations, avoiding any possible circular reasoning. Moreover, we suggest a new way to visualize the distribution of the magnitudes, based on the random variable transformation of Taroni (2023), that switches exponential variables to simple Uniform [0, 1] variables.

METHODS

The exponentiality of the magnitudes can be tested by applying the Lilliefors test to $X_i = M_i - M_c$, where M_i is the i-th magnitude in the catalogue and M_c is the completeness threshold (Lilliefors 1969; Herrmann & Marzocchi 2021). Such a standard test is used to check the exponentiality of any type of data.

Another way to test the exponentiality of the magnitudes is to transform them using the formula (Taroni 2023):

$$U_i = \frac{X_i}{X_i + X_{i+1}}, \tag{2}$$

where X_i are the ones previously described. In this case, coupling the consecutive magnitudes in the catalogue (index i: [1,2], [3,4], [5,6], etc.), the transformation from X_i to U_i allows to move from N

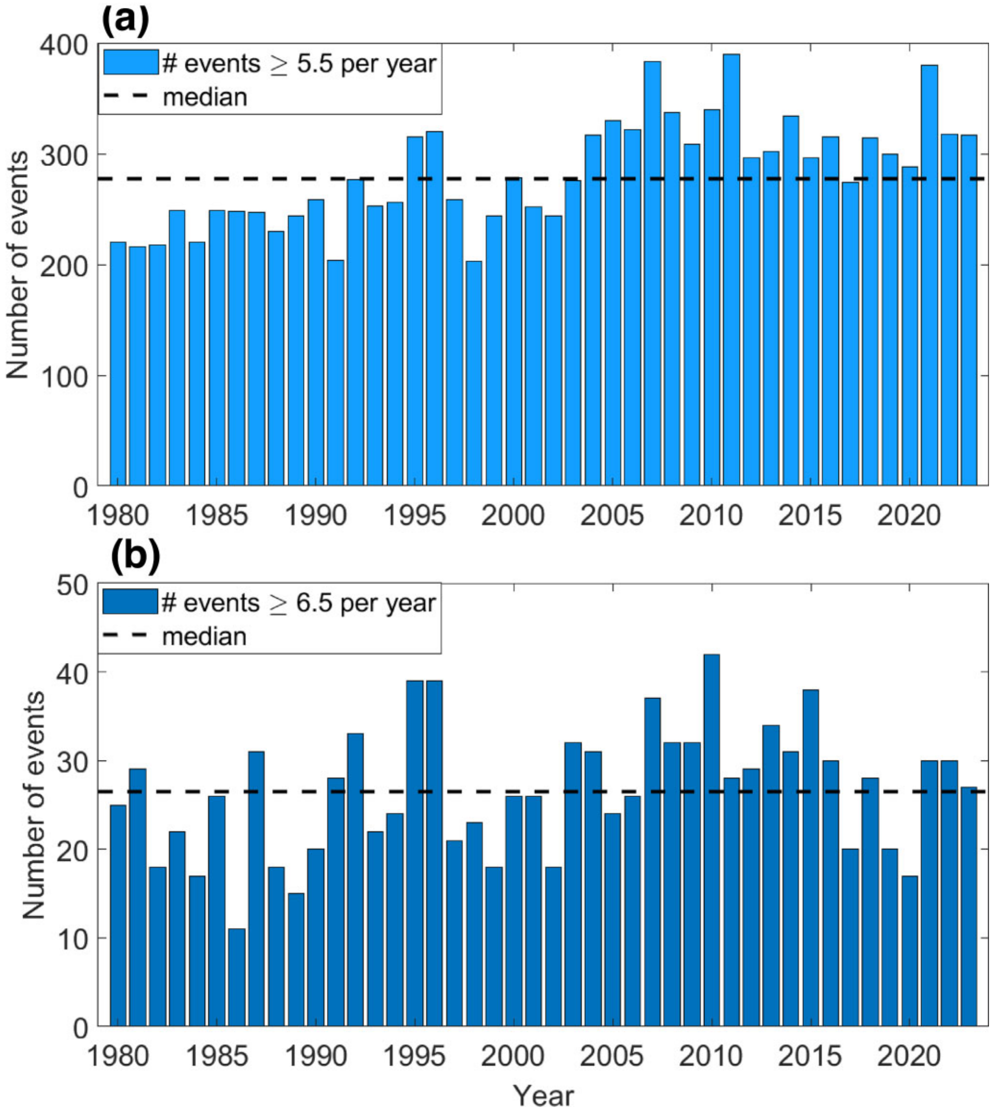

Figure 1. Panel (a): yearly number of events in the catalogue from $M_w = 5.5$, the black dotted line represents the median for all the years. Panel (b): same as (a), but from $M_w = 6.5$.

exponential to $\sim \frac{N}{2}$ uniform variables. The use of non-overlapping indexes (i.e. [1,2], [3,4], [5,6] instead of [1,2], [2,3], [3,4]) allows for easily testing and simulating these variables, avoiding internal correlations. The uniform variables U_i are characterized by a horizontal probability density function (PDF) in their domain [0, 1], and also a linear cumulative distribution function (CDF). After the transformation, we can apply a test of uniformity to these variables U_i, for example, the standard Kolmogorov–Smirnov test (Smirnov 1948).

A set of magnitudes could be considered a random sample from an exponential distribution if it is both exponentially distributed and independent of one from the other (Lombardi & Marzocchi 2007). Two typical tests to check the independence between the magnitudes are the runs test and the test on the correlation coefficient computed on pairs of consecutive magnitudes (Lombardi & Marzocchi 2007; Gibbons & Chakraborti 2010; Taroni 2024). We underline that these tests just check the correlations between consecutive magnitudes, not for all the possible couples; however, they are a sort of standard in statistical seismology (Petrillo & Zhuang 2022), thus we will use them in this paper. To estimate the b-value, we used the Aki (1965) formula with the correction for an unbiased estimation (Ogata & Yamashina 1986):

$$\hat{b} = \frac{\frac{N-1}{N}}{\bar{X}\,\log(10)}, \qquad (3)$$

where N is the number of events above the magnitude of completeness, and $\bar{X}$ is the mean of the X_i. The uncertainty σ on the estimated b-value is $\hat{\sigma}_{\hat{b}} = \frac{\hat{b}}{\sqrt{N}}$ (Aki 1965).

RESULTS AND DISCUSSION

We selected from the global CMT catalogue (Dziewonski *et al.* 1981; Ekström *et al.* 2012) events from 1980 to 2023, with a maximum depth of 50 km (as in Petruccelli *et al.* 2019). To avoid any circular reasoning, the magnitude of completeness was selected without assuming any relation between the magnitudes with the Gutenberg–Richter law. Looking at previous works on the CMT catalogue (Schorlemmer *et al.* 2005; Eberhard *et al.* 2012; Taroni *et al.* 2014; Gonzalez 2024), and considering the yearly number of events and the median of the magnitudes (Kagan 2003; Ekström

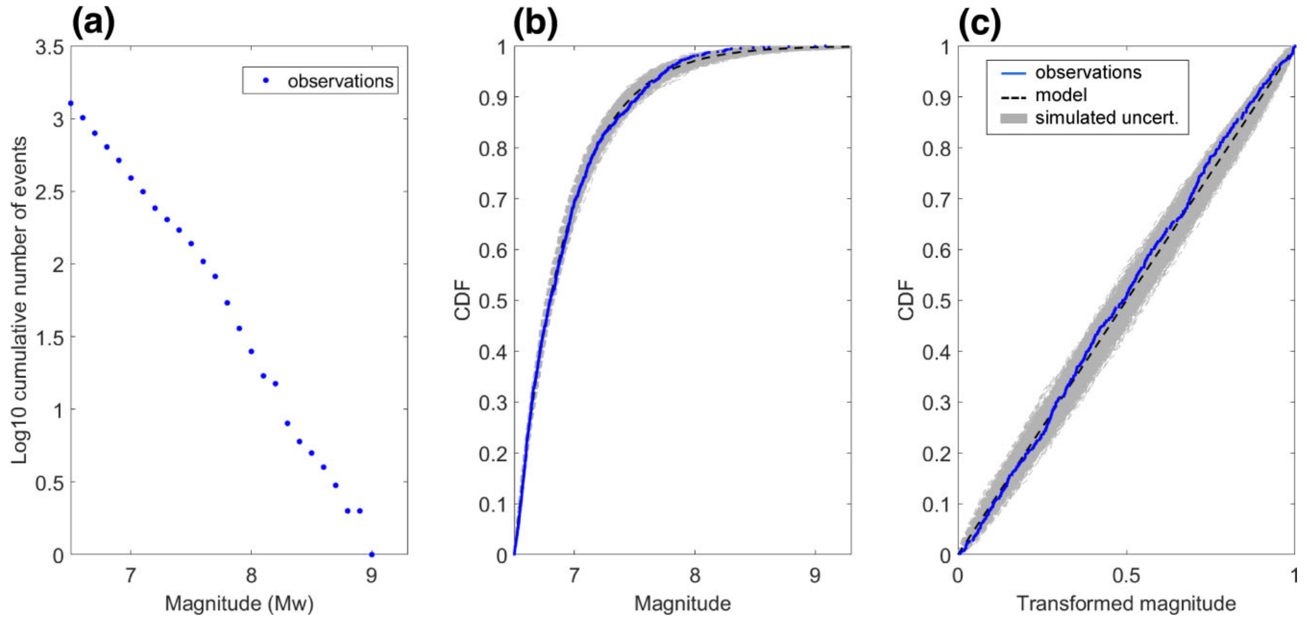

Figure 2. Panel (a): magnitude frequency distribution plot; blue dots represent the log10 cumulative number of events (histogram binning of 0.1). Panel (b): empirical CDF of the magnitudes; blue dots represent the M_i; black dashed line represents the theoretical CDF of the exponential distribution; grey area represents the simulated confidence region for the CDF (with the parameter b-value estimated from the data). Panel (c): empirical CDF of the transformed magnitudes; blue dots represent the U_i; black dashed line represents the theoretical CDF of the Uniform [0, 1] distribution; grey area represents the simulated confidence region for the CDF.

Table 1. p-values of the statistical tests.

Tested feature	Type of variables	Type of test	p-value
Exponentiality	$X_i = M_i - M_c$	Lilliefors test	0.26
Exponentiality	Transformed (U_i)	Uniform test	0.48
Independence	$X_i = M_i - M_c$	Runs test	0.78
Independence	$X_i = M_i - M_c$	Correlation coefficient test	0.30

et al. 2012), the value of $M_w = 6.5$ can be considered a conservative threshold for the magnitude of completeness, since it is larger than any values estimated in the aforementioned papers. In particular, Gonzalez (2024) used the Entire Magnitude Range method (Woessner & Wiemer 2005), finding a decreasing trend in the completeness of the CMT catalogue, from about 5.7 in 1980 to about 5.1 in the last years. The $M_w = 6.5$ value is also larger than any values of the spatially varying completeness estimated by Petrucelli *et al.* (2019) and Gonzalez (2024); however, we have also to take into account the so-called short-term incompleteness problem (Kagan 2003), that is, the low detection capability of a seismic network after strong events, a problem that affects the CMT catalogue (Iwata 2008). Since our catalogue contains several very strong events (25 with magnitude 8.0+), the value of $M_w = 6.5$ seems appropriate to overcome the short-term incompleteness problem. We want to underline that the conservative choice of $M_w = 6.5$ is necessary to ensure the use of a complete catalogue in our calculations.

As a supplementary simple check for completeness, in Fig. 1 we show the yearly number of events from $M_w = 5.5$. (upper panel) and from $M_w = 6.5$ (lower panel). In 2004, the CMT catalogue was improved, leading to a lowering of the event detection threshold (Ekström *et al.* 2012): looking at Fig. 1(a), since the yearly number of events is almost always lower than the median before 2004, we can suspect that the value of $M_w = 5.5$ is too low as a magnitude of completeness of the catalogue. Applying the runs test to the time-series of the yearly number of events, to check its possible randomness, we find a very low p-value of $\sim 10^{-7}$, confirming that for $M_w = 5.5$ the catalogue is not complete.

Different is the case of $M_w = 6.5$ (Fig. 1b), where the observations above and below the median are more equally distributed before and after the 2004, and the p-value of the runs test is 0.05, suggesting the randomness in the order of this time-series and thus the completeness of the catalogue.

By selecting only events with magnitude $M_w \geq 6.5$, our final catalogue contains 1274 earthquakes, a sufficient number of data for robust testing of the Gutenberg–Richter/exponentiality hypothesis for the magnitude distribution. We tested the exponentiality of the magnitudes in two ways: first applying the classical Lilliefors test and then the test of uniformity to the transformed magnitudes (see eq. 2). Using this transformation, it is also possible to build a simple empirical CDF, and visually compare it with the theoretical diagonal line. We underline that, independently from the parameter of the starting exponential distribution of X_i (i.e. the b-value), the transformation will lead to Uniform [0, 1] variables. In Fig. 2 we show both the classical magnitude frequency distribution (Fig. 2a) and the empirical CDFs (Figs 2b and c).

The advantage of the empirical CDFs is that it is possible to easily compare observations and theoretical distributions; in the classical magnitude frequency distribution plot, it is more difficult;

moreover, the points in the left part count a number of events much larger than the points in the right part. Therefore, points in the right part have a large stochastic variability, given the smaller amount of data they represent. As an example, the misalignment of the events with M_w 8.2 + is not so suspicious if we consider that they represent less than 1.5 per cent of the data. Random fluctuations should not be confused with a physical signal (Michael 2011). Moreover, it is quite simple to build a confidence region for the CDF variability through simulations (e.g. the grey area in Figs 2b and c), and then evaluate if the observed variability of the CDF is consistent with random fluctuation, as in the case of Figs 2(b) and (c).

The *p*-values of the tests for the exponentiality are shown in Table 1, along with the tests for the correlations between magnitudes.

The high values of all the *p*-values in Table 1 suggest that the hypotheses of exponentiality and independence cannot be rejected. Thus, our data set can be considered as a random sample from an exponential distribution, confirming the validity of the Gutenberg–Richter law. Using the Aki (1965) formula (eq. 2) we obtained a *b*-value $= 1.02 \pm 0.03$, a value coherent with previous studies (Frohlich & Davis 1993; Kagan *et al.* 2010).

The global catalogue used in this study probably consists of a mixture of exponential distributions with different parameters (i.e. different *b*-values); however, such a mixture does not affect the final exponentiality of the whole catalogue. A possible explanation of this result could be that *b*-values in distinct tectonic regions, or for distinct focal mechanisms, are not that different (Taroni & Carafa 2023).

In this work, we only look at the whole distribution of events, without focusing on the right tail. The right tail of the Gutenberg–Richter law could be described by adding a taper on the exponential distribution (see e.g. Kagan 2002); however, this tail usually contains just a very small percentage of the events in the catalogue and does not affect the exponentiality of the whole distribution.

CONCLUSION

The 44-yr-long worldwide catalogue of M_w 6.5+ events, from 1980 to 2023, containing 1274 earthquakes, confirms the validity of the Gutenberg–Richter law. The magnitudes of these earthquakes are exponentially distributed and independent. This result, although expected, is important because it was obtained without using circular reasoning for the estimation of the magnitude of completeness. The Gutenberg–Richter law is a fundamental part not only of the probabilistic seismic hazard analyses (Gerstenberger *et al.* 2020), but also of short-term earthquake forecasting models (Mizrahi *et al.* 2024). Then, a proper test of its validity is essential to make all these analyses and models scientifically sound.

ACKNOWLEDGMENTS

The author would like to thank the Editor and Reviewers, in particular Andy Michael, for their valuable comments, which significantly improved the quality of this work. This research was supported by the Seismic Hazard Center (Centro di Pericolosità Sismica [CPS]) at the Istituto Nazionale di Geofisica e Vulcanologia (INGV).

SUPPORTING INFORMATION

Supplementary data are available at *GJIRAS* online.

Table S1. *p*-values of the statistical tests for a magnitude of completeness M_w 6.4 (1636 events).

Table S2. *p*-values of the statistical tests for a magnitude of completeness M_w 6.6 (1014 events).

Please note: Oxford University Press is not responsible for the content or functionality of any supporting materials supplied by the authors. Any queries (other than missing material) should be directed to the corresponding author for the paper.

DATA AVAILABILITY

The data and code are freely available at: https://github.com/MatteoTaroniINGV/GutenbergRichter_Testing

REFERENCES

Aki, K. 1965. Maximum likelihood estimate of *b* in the formula log *N*= *a*–*b*M and its confidence limits, *Bull. Earthquake Res. Inst., Tokyo Univ.,* **43,** 237–239.

Bird, P. & Kagan, Y.Y. 2004. Plate-tectonic analysis of shallow seismicity: apparent boundary width, beta, corner magnitude, coupled lithosphere thickness, and coupling in seven tectonic settings, *Bull. seism. Soc. Am.,* **94**(6), 2380–2399.

Dziewonski, A.M., Chou, T.A. & Woodhouse, J.H. 1981. Determination of earthquake source parameters from waveform data for studies of global and regional seismicity, *J. geophys. Res.: Solid Earth,* **86**(B4), 2825–2852.

Eberhard, D.A., Zechar, J.D. & Wiemer, S. 2012. A prospective earthquake forecast experiment in the western Pacific, *Geophys. J. Int.,* **190**(3), 1579–1592.

Ekström, G., Nettles, M. & Dziewoński, A.M. 2012. The global CMT project 2004–2010: centroid-moment tensors for 13,017 earthquakes, *Phys. Earth planet. Inter.,* **200-201,** 1–9.

Frohlich, C. & Davis, S.D. 1993. Teleseismic *b* values; or, much ado about 1.0, *J. geophys. Res.: Solid Earth,* **98**(B1), 631–644.

Gerstenberger, M.C.et al. 2020. Probabilistic seismic hazard analysis at regional and national scales: state of the art and future challenges, *Rev. Geophys.,* **58**(2), e2019RG000653.

Gibbons, J.D. & Chakraborti, S. 2010. *Nonparametric Statistical Inference,* 5th edn, CRC Press.

González, Á. 2024. Improvements and heterogeneities of the Global Centroid Moment Tensor catalog, *Seismol. Res. Lett.,* **95**(6), 3566–3578.

Gutenberg, B. & Richter, C.F. 1944. Frequency of earthquakes in California, *Bull. seism. Soc. Am.,* **34**(4), 185–188.

Herrmann, M. & Marzocchi, W. 2021. Inconsistencies and lurking pitfalls in the magnitude–frequency distribution of high-resolution earthquake catalogs, *Seismol. Res. Lett.,* **92**(2A), 909–922.

Iwata, T. 2008. Low detection capability of global earthquakes after the occurrence of large earthquakes: investigation of the Harvard CMT catalogue, *Geophys. J. Int.,* **174**(3), 849–856.

Kagan, Y.Y. 2002. Seismic moment distribution revisited: I. Statistical results, *Geophys. J. Int.,* **148**(3), 520–541.

Kagan, Y.Y. 2003. Accuracy of modern global earthquake catalogs, *Phys. Earth planet. Inter.,* **135**(2-3), 173–209.

Kagan, Y.Y., Bird, P. & Jackson, D.D. 2010. Earthquake patterns in diverse tectonic zones of the globe, *Pure appl. Geophys.,* **167,** 721–741.

Lilliefors, H.W. 1969. On the Kolmogorov-Smirnov test for the exponential distribution with mean unknown, *J. Am. Stat. Assoc.,* **64**(325), 387–389.

Lombardi, A.M. & Marzocchi, W. 2007. Evidence of clustering and nonstationarity in the time distribution of large worldwide earthquakes, *J. geophys. Res.: Solid Earth,* **112**(B2,). doi:

Michael, A.J. 2011. Random variability explains apparent global clustering of large earthquakes, *Geophys. Res. Lett.,* **38**(21),

Mizrahi, L.et al. 2024. Developing, testing, and communicating earthquake forecasts: current practices and future directions, *Rev. Geophys.,* **62**(3), e2023RG000823.

Ogata, Y. & Katsura, K. 1993. Analysis of temporal and spatial heterogeneity of magnitude frequency distribution inferred from earthquake catalogues, *Geophys. J. Int.,* **113**(3), 727–738.

Ogata, Y. & Yamashina, K.I. 1986. Unbiased estimate for *b*-value of magnitude frequency, *J. Phys. Earth,* **34**(2), 187–194.

Petrillo, G. & Zhuang, J. 2022. The debate on the earthquake magnitude correlations: a meta-analysis, *Sci. Rep.,* **12**(1), 20683.

Petruccelli, A., Schorlemmer, D., Tormann, T., Rinaldi, A.P., Wiemer, S., Gasperini, P. & Vannucci, G. 2019. The influence of faulting style on the size-distribution of global earthquakes, *Earth planet. Sci. Lett.,* **527,** 115791.

Schorlemmer, D., Wiemer, S. & Wyss, M. 2005. Variations in earthquake-size distribution across different stress regimes, *Nature,* **437**(7058), 539–542.

Smirnov, N. 1948. Table for estimating the goodness of fit of empirical distributions, *Ann. Math. Stat.,* **19**(2), 279–281.

Taroni, M. 2023. Estimating the magnitude of completeness of earthquake catalogs using a simple random variable transformation, *The Seismic Record,* **3**(3), 194–199.

Taroni, M. 2024. Are the magnitudes of earthquakes in Southern California, with incompleteness removed, correlated?, *Geophys. J. Int.,* **236**(3), 1596–1600.

Taroni, M. & Carafa, M.M.C. 2023. Earthquake size distributions are slightly different in compression versus extension, *Commun. Earth Environ.,* **4**(1), 398.

Taroni, M., Zechar, J.D. & Marzocchi, W. 2014. Assessing annual global M 6+ seismicity forecasts, *Geophys. J. Int.,* **196**(1), 422–431.

Utsu, T. 1965. A method for determining the value of *b* in a formula log $n = a–b$M showing the magnitude frequency relation for earthquakes, *Geophys. Bull. Hokkaido Univ.,* **13,** 99–103.

Wiemer, S. & Wyss, M. 2000. Minimum magnitude of completeness in earthquake catalogs: examples from Alaska, the western United States, and Japan, *Bull. seism. Soc. Am.,* **90**(4), 859–869.

Woessner, J. & Wiemer, S. 2005. Assessing the quality of earthquake catalogues: estimating the magnitude of completeness and its uncertainty, *Bull. seism. Soc. Am.,* **95**(2), 684–698.